On the Upper Geodetic Cototal Domination Number of a Graph

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ABSTRACT

A geodetic cototal dominating set D in a connected graph G is called a minimal geodetic cototal dominating set of G if no proper subset of D is a geodetic cototal dominating set of G. The maximum cardinality of a minimal geodetic cototal dominating set of G is the upper geodetic cototal domination number of G and is denoted by $\gamma^{+}{}_{gct}(G)$. *It is shown that for every two positive integers* a *and* b *of integers, with* $3 \leq a \leq b$, *there exists a connected graph* G *with* $\gamma_{gct}(G) = a$ *and* $\gamma^{+}{}_{gct}(G) = b$.

Keywords: *geodetic set, cototal domination, geodetic cototal domination, upper geodetic cototal domination.*

AMS Subject Classification: *05C12, 05C69.*

1 Introduction

Graphs are the mathematical structures used to model pairwise relations between objects or a pictorial representation of set of objects where a link connects some pairs of objects. The interacting objects are called points, vertices, or nodes and the relationships that connect the objects are called lines, edges or arcs. A graph $G = (V, E)$, consists of a finite nonempty set V = V(G) of vertices together with a set E = E(G) of un ordered pair e = {u, v} of distinct elements of V. The standard terminology and notations in this article are based on the book Graph theory and Distance in Graphs [1, 2]. In this article, we consider only a finite, undirected graph with no loops or multiple edges. For vertices u and v in a connected graph G, the distance $d(u, v)$ is the length of a shortest $u - v$ path in G. A $u - v$ path of length $d(u, v)$ is called a $u - v$ *geodesic*. The *eccentricity* $e(v)$ of a vertex v in G is the maximum distance from v and a vertex of G. The minimum eccentricity among the vertices of G is the *radius*, $rad\, G$ or $r(G)$ and the maximum eccentricity is its *diameter*, $diam\, G$ of G. Let $x, y \in V$ and let $I[x, y]$ be the set of all vertices that lies in $x - y$ geodesic including x and y. Let $S \subseteq V(G)$ and $I[S] = \cup_{x,y \in S} I[x, y]$. Then S is said to be a *geodetic set* of G, if $I[S] = V$. The *geodetic number* $g(G)$ of G is the minimum order of its geodetic sets and any geodetic set of order $g(G)$ is called a *g-set* of G. A set $S \subseteq V(G)$ is called a *dominating set* if every vertex in $V(G) - S$ is adjacent to at least one vertex of S. The *domination number*, $\gamma(G)$, of a graph

G denotes the minimum cardinality of such dominating sets of G. A minimum dominating set of a graph G is hence often called as a γ-set of G. For the fundamentals of domination concept refer [3]. A dominating set S of G is a *cototal dominating set* if every vertex $v \in V \setminus S$ is not an isolated vertex in the induced subgraph $\langle V \setminus S \rangle$. The cototal domination number $\gamma_{ct}(G)$ of G is the minimum cardinality of a cototal dominating set [4]. A set $S \subseteq V$ is said to be a geodetic cototal dominating set of G, If S is both geodetic set and cototal dominating set of G. The geodetic cototal domination number of G is the minimum cardinality among all geodetic cototal dominating sets in G and denoted by $\gamma_{gct}(G)$. A geodetic cototal dominating set of minimum cardinality is called the γ_{gct}-set of G [5]. The several concepts of geodetic cototal domination number of a graph were studied [6, 7]. The following theorem is used in the sequel.

Theorem 1.1. [6] Each extreme vertex of a connected graph belongs to every geodetic cototal dominating set of G.

2 The upper geodetic cototal domination number of a graph

Definition 2.1. A geodetic cototal dominating set $D \subset V$ is said to be a minimal geodetic cototal dominating set, if there does not exist a set $N \subset D$ that is a geodetic cototal dominating set of the graph G. The upper geodetic cototal dominating number $\gamma_{gct}^{+}(G)$ is the maximum size of a minimal geodetic cototal dominating set of G.

Example 2.2. For the graph G given in Figure .1, $S_1 = \{v_2, v_5\}, S_2 = \{v_2, v_4, v_6\}$ are the only two minimal geodetic cototal dominating sets of G and so $\gamma_{gct}^{+}(G) \geq 3$. Since there is no minimal geodetic cototal dominating set of G with cardinality four, $\gamma_{gct}^{+}(G) = 3$.

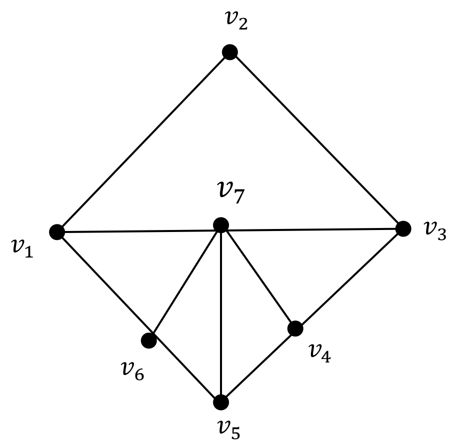

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Fig 1.

Remark 2.3. Every minimum geodetic cototal dominating set of G is a minimal geodetic cototal dominating set of G, but the converse need not be true. For the graph G given in Fig.1, $S_2 = \{ v_2, v_4, v_6\}$ is a minimal geodetic cototal dominating set of G but not a minimum geodetic cototal dominating set of G.

Theorem 2.4. Let G be a connected graph of order n. Then $2 \leq \gamma_{gct}(G) \leq \gamma_{gct}^{+}(G) \leq n$.

Proof: Since every geodetic cototal dominating set of G needs at least two vertices, $\gamma_{gct}(G) \geq 2$. **.** Since every minimum geodetic cototal dominating set of G is a minimal geodetic cototal dominating set of G, it follows that $\gamma_{gct}(G) \leq \gamma_{gct}^{+}(G)$. Also, since V(G) is a geodetic cototal dominating set of G, We have $\gamma_{gct}^{+}(G) \leq n$. Therefore $2 \leq \gamma_{gct}(G) \leq \gamma_{gct}^{+}(G) \leq n$.

Remark 2.5. The bounds in Theorem 2.4 are sharp. For the graph G = K_2, $\gamma_{gct}(G)$=2 = n and for the graph G = C_4 , $\gamma_{gct}(G) = \gamma_{gct}^{+}(G) = 4$. For G = K_n , $\gamma_{gct}^{+}(G) = n$ **.** Also, the bounds in Theorem 2.4 are strict. For the graph in Fig.1, $\gamma_{gct}(G) = 2,\ \gamma_{gct}^{+}(G) = 5$ and n = 7. Thus $2 < \gamma_{gct}(G) < \gamma_{gct}^{+}(G) < n$.

Observation 2.6. (i) For $G = K_n\ (n \geq 2), \gamma_{gct}^{+}(G) = n$.

(ii) For $G = P_n\ (n \geq 5), \gamma_{gct}^{+}(G) = \left\lceil \frac{n}{3} \right\rceil$.

(iii) For $G = C_n\ (n \geq 6), \gamma_{gct}^{+}(G) = \left\lceil \frac{n}{3} \right\rceil$.

(iv) For $G = K_{1,n-1},\ \gamma_{gct}^{+}(G)$ = n.

Theorem 2.7. For G=$K_{r,s}(G) = \begin{cases} r+s, & if\ 1 \leq r \leq 3 \\ 4, & if\ 4 \leq r \leq s \end{cases}$.

Proof: Let $U = \{u_1, u_2, \ldots, u_r\}$ and $W = \{w_1, w_2, \ldots, w_s\}$ be the two bipartite sets of G. Let S be a γ_{gct}-set of G. If $1 \leq r \leq 3$. Then $S = V(G)$ is the unique γ_{gct}-set of G. so that $\gamma_{gct}^{+}(G)$ = 3. So, let $4 \leq r \leq s$. Let $S = \{u_1, u_2, w_1, w_2\}$. Then S is a minimal geodetic cototal dominating set of G and so $\gamma_{gct}^{+}(G) \geq 4$. We prove that S is a minimal geodetic cototal dominating set of G. Suppose this is not the case. Then there exists geodetic cototal dominating of S_1 such that $S_1 \subset S$. Then there exists a vertex $x \in S$ such that $x \notin S_1$. If $x = x_1\ or\ x_2$, then w_1, w_2 do not belong to a geodesic joining a pair of vertices of S. If $x = w_1$or w_2, then u_1, u_2do not belong to a geodesic joining a pair of vertices of S. Therefore S_1 is not a geodetic cototal dominating set of G, which is a contradiction. Hence it follows that S is a minimal geodetic cototal dominating set of G and so that $\gamma_{gct}^{+}(G) \geq 4$. We prove that $\gamma_{gct}^{+}(G)$ = 4. On the contrary suppose that $\gamma_{gct}^{+}(G) \geq 5$, Then there exists a minimal geodetic cototal dominating set S' , such that $|S'| \geq 5$. Since $G[V - S']$ has no isolated vertices, $S' \subset U \cup W$ such that S' contains at least two vertices from U and at least two vertices from W. Hence it follows that there exists a geodetic cototal dominating set M such that $M \subset S'$, which is a contradiction. Therefore $\gamma^{+}_{gct}(G) = 4$.

Definition 2.8. Let $V(\overline{K}_2) = \{x, y\}$ and $(\overline{K}_a) = \{v_1, v_2, \ldots, v_a\}(a \geq 2)$. Let $H = \overline{K}_a + \overline{K}_2$. Let G_a be the graph in Fig. 2 obtained from H by adding a new vertex u and joining u with y.

Theorem 2.9. For the graph G_a, $\gamma_{gct}(G_a) = 2$.

Proof: Let $Z = \{u\}$ be the set of all end vertices of G. By Theorem 1.1, Z is a subset of every geodetic cototal dominating set of G. Let $S = Z \cup \{x\}$. Then S is a geodetic cototal dominating set of G so that $\gamma_{gct}(G_a) = 2$.

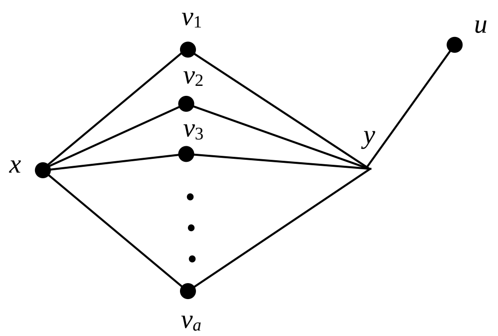

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Fig 2.

Definition 2.10. Consider the paths $P_{2a+1}: x_1, x_2, \ldots, x_{2a+1}$. Let $V(\overline{K}_2)) = \{x, y\}$. Obtain a graph H_a given in Fig. 3 from P_{2a+1} and $\overline{K}_2$ by introducing new vertices $y_1, y_2, \ldots, y_{a-1}$ and edges xx_i $(1 \leq i \leq 2a + 1)$, yx_i $(1 \leq i \leq 2a + 1)$ and yy_i $(1 \leq i \leq a - 1)$.

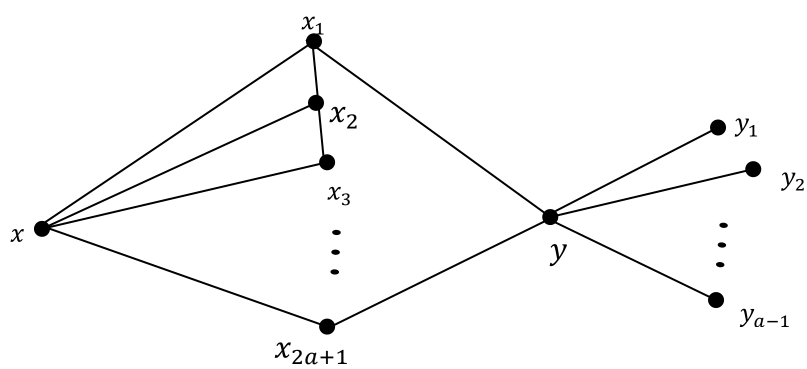

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Fig 3.

Theorem 2.11. For the graph H_a, $\gamma_{gct}(H_a) = a$ and $\gamma^+_{gct}(G) = 2a - 1$.

Proof: Let S be a $\gamma_{ct} - set$ of G and Y be the set of all pendant vertices of G. By Theorem 1.1, $Y\ S. Hence\ \gamma_{gct}(G) \geq a$. Since the vertex $x \notin I[Y]$. Y is not a geodetic cototal dominating set of G and so $\gamma_{gct}(G) \geq a$. Let $S = Y \cup \{x\}$. Then S is a geodetic cototal dominating set of G so that $\gamma_{gct}(G) = a$.

Next, we have to prove $\gamma^+_{gct}(G)$= $2a - 1$. Let $S' = Y \cup \{x_1, x_3, \ldots, x_{2a+1}\}$. Then S is a geodetic cototal dominating set of G. We prove that S is a minimal geodetic cototal dominating set of G. Then$S_1 \subseteq S$, where S_1 is a geodetic cototal dominating set of G such that $S_1 \subset S$. Let x be a vertex of S such that $x \notin S_1$. By Theorem 1.1, $x \neq y_i$ $(1 \leq i \leq a - 1)$. If $h = x_i$ $(1 \leq i \leq 2a + 1)$ then $x_i \notin I[S_1]$, which is a contradiction. Therefore S is a minimal geodetic cototal dominating set of G so that $\gamma^+_{gct}(G) \geq 2a - 1$.

We prove that $\gamma^+_{gct}(G) = 2a - 1$. On the contrary, suppose that $\gamma^+_{gct}(G) \geq 2a - 1$. Then there exists a minimal geodetic cototal dominating set M such that $| M | \geq b + 1$. By Theorem 1.1, $Y \subset M$, which is a contradiction. Therefore $\gamma^+_{gct}(G) = 2a - 1$.

Theorem 2.12. For any connected graph G of order $n \geq 2$. Then $\gamma_{gct}(G) = n$ if and only if $\gamma^+_{gct}(G) = n$.

Proof: If $\gamma_{gct}(G) = n$. Then by Theorem 2.5, $\gamma^+_{gct}(G) = n$. Conversely let $\gamma^+_{gct}(G) = n$. Then $S = V(G)$ is the unique minimal geodetic cototal dominating set and so it is a minimum geodetic cototal dominating set of G. Hence $\gamma_{gct}(G) = n$.

Theorem 2.13. For every pair of a, b with $2 \leq a \leq b$ and $b \geq 2$, there exists a connected graph G such that $\gamma_{gct}(G)$ = a and $\gamma^+_{gct}(G) = b$.

Proof: **C**onsider the paths $P : u_0, u_1, u_2$ and $Q : h_1, h_2, \ldots, h_{b-a+2}$ .From P and Q obtain a new graph H by joining u_1 with h_1 and u_0 and u_2 with each h_i $(1 \leq i \leq b - a + 2)$. From H obtain a new graph G given in Fig.4. by introducing the vertices $z_1, z_2, \ldots, z_{a-2}$ and introducing the edge $u_i x_i$ $(1 \leq i \leq a - 2)$.

We prove that $\gamma_{gct}(G) = a$. The graph having set of pendent vertices $Z = \{z_1, z_2, \ldots, z_{a-2}\}$. Then by Theorem 1.1, Z is contained in every geodetic cototal dominating set of G. Since $I[Z] \neq V$ and $I[Z \cup \{x\}]$, where $x \notin Z$ is not a geodetic cototal dominating set of G and so $\gamma_{gct}(G) \geq a$. Now $S = Z \cup \{u_0, u_2\}$ is a geodetic cototal dominating set of G.Therefore $\gamma_{gct}(G) = a$.

Next, we have to prove $\gamma^+_{gct}(G) = b$. Consider $S = Z \cup \{h_1, h_2, \ldots, h_{b-a+2}\}$. Then S is a geodetic cototal dominating set of G. We prove that S is a minimal geodetic cototal dominating set of G. Suppose this is not the case. Hence $S_1 \subset S$, where S_1 is a minimal geodetic cototal dominating set of G. Let x be a vertex of S such that $x \notin S_1$. By Theorem 1.1, $x \neq z_i\ (1 \le i \le a + 2)$. If $x = h_i\ (1 \le i \le b - a + 2)$ then $x \notin I[S_1]$, which is a contradiction. Therefore, S is a minimal geodetic cototal dominating of G so that $\gamma^+_{gct}(G) \ge b$. We prove that $\gamma^+_{gct}(G) = b$. On the contrary, suppose that $\gamma^+_{gct}(G) \ge$ b + 1. Then there exists a geodetic cototal dominating set M such that $|M| \ge b + 1$. By Theorem 1.1, $Z \subset M$. Since S is a minimal geodetic cototal dominating set of G and $u_1 \notin M$, there does not exists a minimal geodetic cototal dominating with $|M| \ge b + 1$. Therefore $\gamma^+_{gct}(G) = b$.

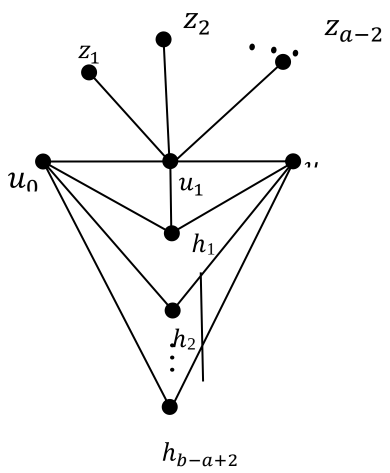

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Fig 4.

Conclusion

In this paper the concept of upper geodetic cototal domination number of some standard graphs some general properties satisfied by this concept are studied. In future studies, the same concept will be applied for the other graph operations.

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